Semiclassical quantization with short periodic orbits

Eduardo G. Vergini and Gabriel Carlo
Departamento de Física, Comisión Nacional de Energía Atómica. Av. del Libertador 8250, 1429 Buenos Aires, Argentina.
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We apply a recently developed semiclassical theory of short peridic orbits to the stadium billiard. We give explicit expresions for the resonances of periodic orbits and for the application of the semiclassical Hamiltonian operator to them. Then, by using the 3 shortest periodic orbits and 2 more living in the bouncing ball region, we obtain the first 25 odd-odd eigenfunctions with surprising accuracy.

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The study of semiclassical tecniques in order to obtain quantum information of classically chaotic Hamiltonian systems, has received much attention in the last 30 years [1–7]; most of the developed methods are related to the Gutzwiller's trace formula [1]. This formula is very attractive because gives the energy spectrum of a bounded system in terms of periodic orbits (**PO**s). However, the number of **PO**s required in the calculuation is enormous and increases exponentially with the Heisenberg time $T_H \equiv 2\pi\hbar\rho_E$ (ρ_E is the mean energy density).

Recently, a new approach has been developed [8] which uses a very little number of short **PO**s in the chaotic region. In order to verify the power of this new formalism, we applied it to the Bunimovich stadium billiard with radius $R = 1$ and straigth line 2, an ergodic system [9]. The starting point is the construction of resonances of a given unstable **PO**. They are functions (highly localized in energy) living in a neighborhood of the **PO**. We will give the classical elements in order to obtain explicit expresions for the resonances and for the application of the semiclassical Hamiltonian operator to them. Then, we select a set of resonances such that its mean density agrees with semiclassical prescriptions. Finally, we will evaluate eigenfunctions and eigenvalues of the billiard by solving a generalized eigenvalue problem.

Let γ be a **PO** of the desymmetrized stadium billiard with turning points (a libration [10]); see Fig. 1. Let x be the coordinate along γ with the origin $x = 0$ at one of the turning points; the other being at $x = L/2$, with L the length of γ. The transversal coordinate is y, with $y = 0$ on γ.

Let $M(x)$ be a symplectic matrix describing the linearized transversal motion along γ; that is, a point with transversal coordinates (y, p_y) at $x = 0$ evolves according to the following rule: $(y(x), p_y(x)) = (y, p_y)M(x)^t$. Then, $\tilde{M}(x) \equiv (-1)^{N(x)}M(x)$, with $N(x)$ the number of bounces with the desymmetrized boundary while evolving from 0 to x, is obtained with two types of matrices:

$$M_1(l) = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_2(\theta) = \begin{pmatrix} 1 & 0 \\ -2/\cos(\theta) & 1 \end{pmatrix}.$$

$M_1(l)$ describes the evolution for a path of length l without bounces with the circle (the transversal momentum is measured in units of the momentum along the trajectory), and $M_2(\theta)$ has into account a bounce with the circle (θ defines the angle between the incoming trajectory and the radial direction). $\sqrt{M_2(\theta)}$ is obtained from $M_2(\theta)$ by replacing the 2 by 1.

If the point $x = 0$ (or $x = L/2$) is over the circle, we divide the contribution $M_2(\theta)$ given by the bounce, between the incoming and outgoing path. For example (see Fig. 1), $\tilde{M}(L/2) = M_1(\sqrt{5})\sqrt{M_2(\theta)}$ for orbit (a), $M_1(1+\sqrt{2})\sqrt{M_2(0)}$ for (b), $M_1(\sqrt{3})M_2(\pi/6)M_1(\sqrt{3}/2)$ for (c), $\sqrt{M_2(0)}M_1(1+\sqrt{5})$ for (d), $\sqrt{M_2(0)}M_1(1+\sqrt{10})$ for (e), and $M_1(1/\sqrt{2})M_2(\pi/4)M_1(1+1/\sqrt{2})$ for (f).

By using time reversal, it is easy to see that

$$\tilde{M}(L) = \begin{pmatrix} d & b \\ c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the explicit matrix on the right is $\tilde{M}(L/2)$ and the other completes the orbit (from $x = L/2$ to $x = L \equiv 0$). Moreover, as the diagonal elements of $\tilde{M}(L)$ are equal, the matrix can be written as follows [8]

$$\tilde{M}(L) = (-1)^{\nu} \begin{pmatrix} \cosh(\lambda L) & \sinh(\lambda L)/\tan(\varphi) \\ \sinh(\lambda L)\ \tan(\varphi) & \cosh(\lambda L) \end{pmatrix}.$$

$\lambda = (1/L)\ln(|A| + \sqrt{A^2 - 1})$ ($A \equiv ad + bc$) is the Lyapunov exponent in units of $[length^{-1}]$, $\tan(\varphi)$ ($\neq 0$) in units of $[length^{-1}]$ defines the slope of the unstable manifold in the plane $y - p_y$ (the slope of the stable manifold being $-\tan(\varphi)$), and ν is the maximum number of conjugated points along γ. Finally, being ξ_u and ξ_s the unstable and stable directions respectively, the symplectic matrix B transforming coordinates from the new directions into the old ones (y and p_y) is

$$B = (\xi_u\ \xi_s) = (1/\sqrt{2}) \begin{pmatrix} 1/\alpha & -s/\alpha \\ s\,\alpha & \alpha \end{pmatrix},$$

with $\alpha \equiv \sqrt{|\tan(\varphi)|} = |ac/bd|^{1/4}$, and $s \equiv \text{sign}(\varphi) = \text{sign}(acA)$.

Now, it is possible to construct a family of resonances associated to γ. Resonances within a family are identified by $n = 0, 1, \ldots$, the number of excitations along the trajectory, and the wave number k used in the construction depends on γ and n through the Bohr-Sommerfeld quantization rule:

$$Lk - (N_s + s_h N_h + s_v N_v)\pi - \nu\pi/2 = 2n\pi.$$

ν is equal to the number of bounces with the circle, N_s with the stadium boundary, and N_h (N_v) with the horizontal (vertical) symmetry line. $s_h = 0$ (1) for even (odd) symmetry on the horizontal axis and equivalently with s_v for the vertical axis. $(\nu, N_s, N_h, N_v) = (1,2,1,1)$ for (a), $(1,2,2,1)$ for (b), $(2,2,2,1)$ for (c), $(1,3,3,1)$ for (d), $(1,4,4,1)$ for (e), and $(2,2,1,1)$ for (f).

Resonances are constructed with straight lines by associating a semiclassical expression to each one. The first line is defined by the segment of γ starting at $x_1 = 0$. Let x_2 ($> x_1$) be the value of x such that the path reaches the stadium boundary while evolving along γ. The path going out of x_2 defines the second line, and so on up to $x = L/2$. It is necessary 1 line for (a) and (b), and 2 lines for (c), (d), (e) and (f) (see Fig. 1).

Defining local coordinates $(x^{(j)}, y^{(j)})$ on each line such that $x^{(j)} = x$ inside the desymmetrized billiard, the expresion for line j is [in the following expresions we are going to use (x, y) understanding $(x^{(j)}, y^{(j)})$]

$$\psi_j(x,y) = f_j(x,y)\ \sin[ky^2 g_j(x) + kx - \Phi_j(x)], \qquad (1)$$

with $g_j(x) = [Q_j^*(x)P_j(x) + Q_j(x)P_j^*(x)]/(4\ |Q_j(x)|^2)$, $f_j(x,y) = 2(k/\pi)^{1/4}\exp[-ky^2/(2\ |Q_j(x)|^2)]/\sqrt{L|Q_j(x)|}$, and $\Phi_j(x) = \pi N_D(x_j^+) + \varphi_j + [\phi_j(x) + \alpha_j(x)]/2$. $N_D(x_j^+)$ is the number of bounces up to x_j (including the bounce at x_j) satisfying Dirichlet boundary conditions; $N_D(x_1^+) = 0$. Moreover,

$$\begin{pmatrix} Q_j(x) \\ P_j(x) \end{pmatrix} = M_1(x - x_j) \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} e^{-(x-x_j)\lambda} \\ i\ e^{(x-x_j)\lambda} \end{pmatrix},$$

with

$$\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} = \tilde{M}(x_j^+) B \begin{pmatrix} e^{-x_j\lambda} & 0 \\ 0 & e^{x_j\lambda} \end{pmatrix}.$$

x_j^+ means (for $j \geq 2$) that $\tilde{M}$ is evaluated after the bounce with the boundary at x_j [$\tilde{M}(x_2^+) = M_2(\pi/6)M_1(\sqrt{3}/2)$ for orbit (c), $M_1(\sqrt{5}/2)$ for (d), $M_1(2\sqrt{10}/3)$ for (e), and $M_2(\pi/4)M_1(1 + 1/\sqrt{2})$ for (f)]. $\tilde{M}(x_1^+) = \sqrt{M_2(\theta)}$ if x_1 lies on the circle; otherwise $\tilde{M}(x_1^+) = \mathbb{1}$. $\phi_j(x) = \arg[Q_j(x)] - \arg[Q_j(x_j)]$ (arg takes the argument of a complex number in the range $[0, 2\pi)$). $\alpha_j(x) = 2\pi\ \mathrm{sign}(x - x_j)$ if $n_j(x) \neq n_j(x_j)$ and $(x - x_j)\phi_j(x) < 0$; otherwise $\alpha_j(x) = 0$, where

$$n_j(x) = \begin{cases} 1 & \text{if } x - x_j > \max(-a_j/c_j, -b_j/d_j) \\ -1 & \text{if } x - x_j < \min(-a_j/c_j, -b_j/d_j) \\ 0 & \text{otherwise.} \end{cases}$$

If $c_j = 0$ or $d_j = 0$, x_j is replaced by any other point on line j, inside the desymmetrized billiard. $\phi_j(x) + \alpha_j(x)$ defines the angle swept by $Q_j(x^{(j)})$ in a continuous way. Finally, $\varphi_j = \varphi_{j-1} + [\phi_{j-1}(x_j) + \alpha_{j-1}(x_j)]/2$ for $j \geq 2$. The value of φ_1 depends on the starting point and the symmetry, and all the possibilities are considered in Fig. 1. φ_1 is equal to $-s_h\pi/2$ for (a), 0 for (b), $(s_h - 1)\pi/2$ for (c), $(s_h + s_v - 1)\pi/2$ for (d), $-s_v\pi/2$ for (e), and $(s_v - 1)\pi/2$ for (f).

The transformation from local coordinates $(x^{(j)}, y^{(j)})$ on line j to coordinates (X, Y) (horizontal and vertical directions in the plane respectively) is obtained through a simple transformation. If (X_j, Y_j) are the coordinates of the point x_j, and α_j the angle of line j with the horizontal direction, $(x^{(j)} - x_j, y^{(j)}) = G_j(X, Y)$ is given by

$$G_j(X,Y) = (X - X_j, Y - Y_j) \begin{pmatrix} \cos(\alpha_j) & -\sin(\alpha_j) \\ \sin(\alpha_j) & \cos(\alpha_j) \end{pmatrix}.$$

Finally, the family of resonances $\psi_\gamma(X, Y)$ is constructed with all the lines including symmetries (see Fig. 2)

$$\psi_\gamma = \sum_j \sum_{i=1}^{m_h} \sum_{l=1}^{m_v} h_i\ v_l\ \psi_j[(x_j, 0) + G_j(s_l X, s_i Y)]. \qquad (2)$$

$s_i \equiv (-1)^{i+1}$ and $s_l \equiv (-1)^{l+1}$. $h_i = [\delta_{i,1} + \delta_{i,2}(1 - 2s_h)]$ and $v_l = [\delta_{l,1} + \delta_{l,2}(1 - 2s_v)]$. m_h and m_v depend on j and are specified as follows: $m_h = 1$ (2) if the line is (is not) symmetric with respect to the horizontal axis, and equivalently with m_v for the vertical axis; however, $m_h = 2$ and $m_v = 1$ if the line goes through the origin.

We define $\hat{H} \equiv -\nabla^2$ and $E \equiv k^2$ (remember that k depends on γ and n). Then, the semiclassical approximation for $(\hat{H} - E)\psi_\gamma(X, Y)$ is obtained directly from Eq. (2) having into account the following semiclassical prescription [8]

$$(\hat{H} - E)\psi_j(x,y) = \tilde{f}_j(x,y)\ \sin[ky^2 g_j(x) + \Delta(x)], \qquad (3)$$

with $\tilde{f}_j(x,y) = \lambda k\ (2ky^2/|Q_j(x)|^2 - 1)f_j(x,y)$, and $\Delta(x) = kx - \Phi_j(x) + \pi/2 - 2\arg[Q_j(x)]$. That is, the action of $\hat{H} - E$ on ψ_j excites the transversal direction with two excitations. Using these expresions it is possible to obtain matrix elements by direct integration on the domain (the quarter of billiard in this case) [11].

In the semiclassical limit ($n \to \infty$), the following diagonal matrix elements are obtained explicitly [8] (using Dirac's notation): *i)* $\langle\gamma|\gamma\rangle \to 1$, *ii)* $\overline{E} \equiv \langle\gamma|\hat{H}|\gamma\rangle/\langle\gamma|\gamma\rangle \to E$, and *iii)* $\sigma^2 \equiv \langle\gamma|\hat{H}^2|\gamma\rangle/\langle\gamma|\gamma\rangle - \overline{E}^2 \to 2\lambda^2k^2$, with σ the dispersion of ψ_γ. On the other hand, as the operator $\hat{H}$ is not exactly Hermitian, it is defined a symmetrized interaction between **PO**s as follows: $\langle\delta|\hat{H}|\gamma\rangle \equiv (\langle\delta|\hat{H}\gamma\rangle + \langle\gamma|\hat{H}\delta\rangle^*)/2$.

Now we will select the set of resonances defining an appropiate basis. From now on, we will restrict to odd symmetry on the horizontal and vertical directions. For the selection of bouncing ball functions we use Tanner's prescription [12]. That is, for a given number M ($M = 1, 2, \ldots$) of vertical excitations, there are $N = [\sqrt{2M+1} + 5/4]$ horizontal excitations in the range $M < k/\pi < M + 1$. This translates into a number n of longitudinal excitations (along the selected orbit) that is

specified by $n=(M-1)K+N-1$, where K is the number of segments of the trajectory inside the desymmetrized billiard [for instance, $K=2$ for (b)]. Then, for the construction of the function (M,N) we select the trajectory (among (b), (d) and (e)) which gives the resonance with smallest dispersion σ.

The number of bouncing ball wavefunctions is only a fraction of the mean number of states specified by the Weyl's law. Then, using the shortest periodic orbits (a), (b) and (c), we put as many resonances as necessary to obtain the required number. As the period of the three **PO**s is comparable, we select first orbit (c) because the associated resonances have smallest dispersion. Figure 3 clarifies the situation. The first column shows the spectrum of bouncing ball resonances. Over each line appears the label corresponding to Fig. 4(a), M, N, the orbit used, and n. Second column shows the spectrum of resonances constructed with orbit (c), and over each line appears the label and n. The same is applicable to the third and fourth columns with orbits (a) and (b) respectively. In this way, the density of resonances agrees with the semiclassical mean density; however for $k>15$ more orbits are required. Note that orbit (b) is used for the construction of resonances living in the bouncing ball and chaotic regions. This is because the bouncing ball region decreases as $1/\sqrt{k}$. Orbit (f) is not considered because the associated resonances do not satisfy boundary conditions with sufficient accuracy for low energies.

Semiclassical eigenfunctions are constructed with this set of wavefunctions. Of course, only a limited number of them are required for a particular eigenfunction. State 21 (see Fig. 4(b)) needs resonances from 18 to 23 (Fig. 4(a)). The other states use less resonances. Suppose that at k_0 there is an eigenstate of the system; it is constructed with all resonances satisfying $|k-k_0| \leq 0.8$ [8]. Despite this is a semiclassical criterium, it works in general at low energies too. It says that the number N_r of resonances contributing to each eigenstate increases as follows: $N_r \simeq 0.5\, k_0$.

Figure 4(a) shows linear density plots of the 27 selected resonances arranged by energy. Numbers below each plot are the label (left), the squared root of the mean energy, and the dispersion σ in units of the mean energy spacing. Using this basis of functions, we evaluate the overlaps and the Hamiltonian matrix elements using the semiclassical prescription given in (3). Then, by solving a generalized eigenvalue problem, the semiclassical set of eigenfunctions shown in Fig. 4(b) is obtained. Numbers below are the label, the semiclassical wave number and the overlap with the exact solutions, which are displayed in Fig. 4(c). The mean standard deviation of the semiclassical eigenvalues is a fraction (0.06) of the mean level spacing in accordance with the theory. Recent results obtained for the hyperbola billiard [13] show a standard deviation of 0.047 (in units of the mean level spacing) for the first 24 even eigenvalues. Those results were calculated using trace-formula-type techniques, involving 38 131 periodic orbits (193 695 pseudo-orbits), in contrast with the only 5 used in the present work. On the other hand, overlaps of the semiclassical eigenfunctions with the exact ones are surprisingly good (see Fig 4(b)). Moreover, to our knowledge, this is the first semiclassical evaluation of a set of eigenfunctions in a chaotic system using periodic orbits.

In conclusion we have applied successfully the theory of short **PO**s to the stadium billiard. This shows that the classical information contained in short **PO**s is sufficient for obtaining the stationary states of a bounded chaotic Hamiltonian system.

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FIG. 1. Set of periodic orbits of the desymmetrized stadium billiard used (with the exception of orbit (f)) for the construction of resonances. (1) and (2) labels the corresponding straight line.

FIG. 2. Set of lines including symmetries used for the construction of resonances associated to orbit (d). The different sets of coordinates used are indicated.

FIG. 3. Spectrum of resonances used for the evaluation of semiclassical eigenfunctions. The meaning of each column is as follows: the first number over the lines is the label. In the first column, showing the spectrum of bouncing ball resonances, M, N and n are indicated. Letters correspond to the orbit from which they were constructed. In the other three columns only the label and the number n is over the lines; the corresponding orbit, the same for the entire column, is at the bottom of the figure.

FIG. 4. Linear density plots of: (a) the selected resonances. The numbers below each plot are (from left to right) the label, the squared root of the mean energy, and the dispersion σ in units of the mean energy spacing. (b) the semiclassical eigenfunctions. Here, after the label, the semiclassical wave number and the overlap with the corresponding exact eigenfunction are displayed. (c) exact eigenfunctions with their wavenumbers.

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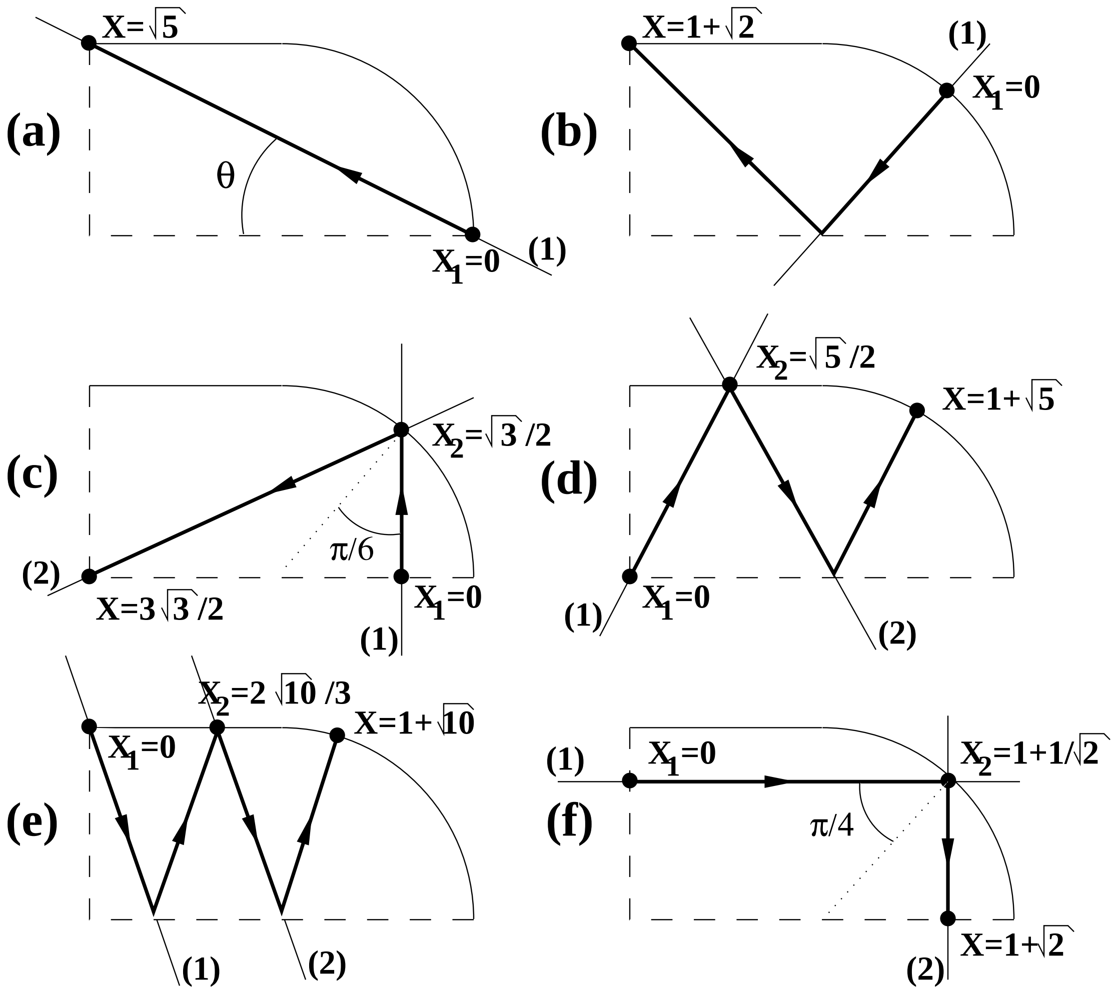
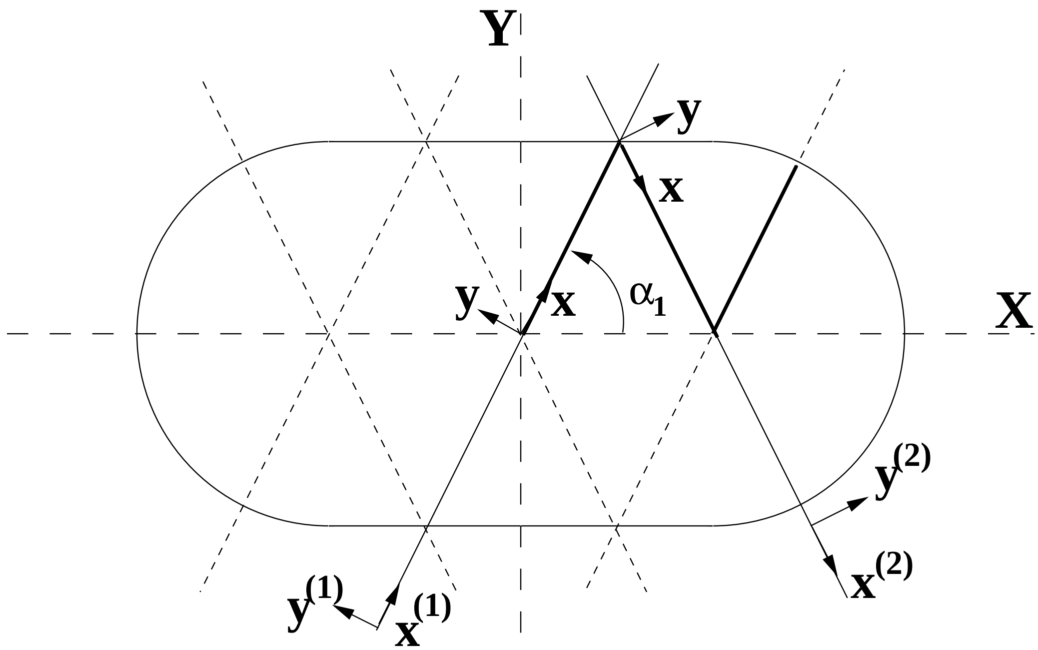
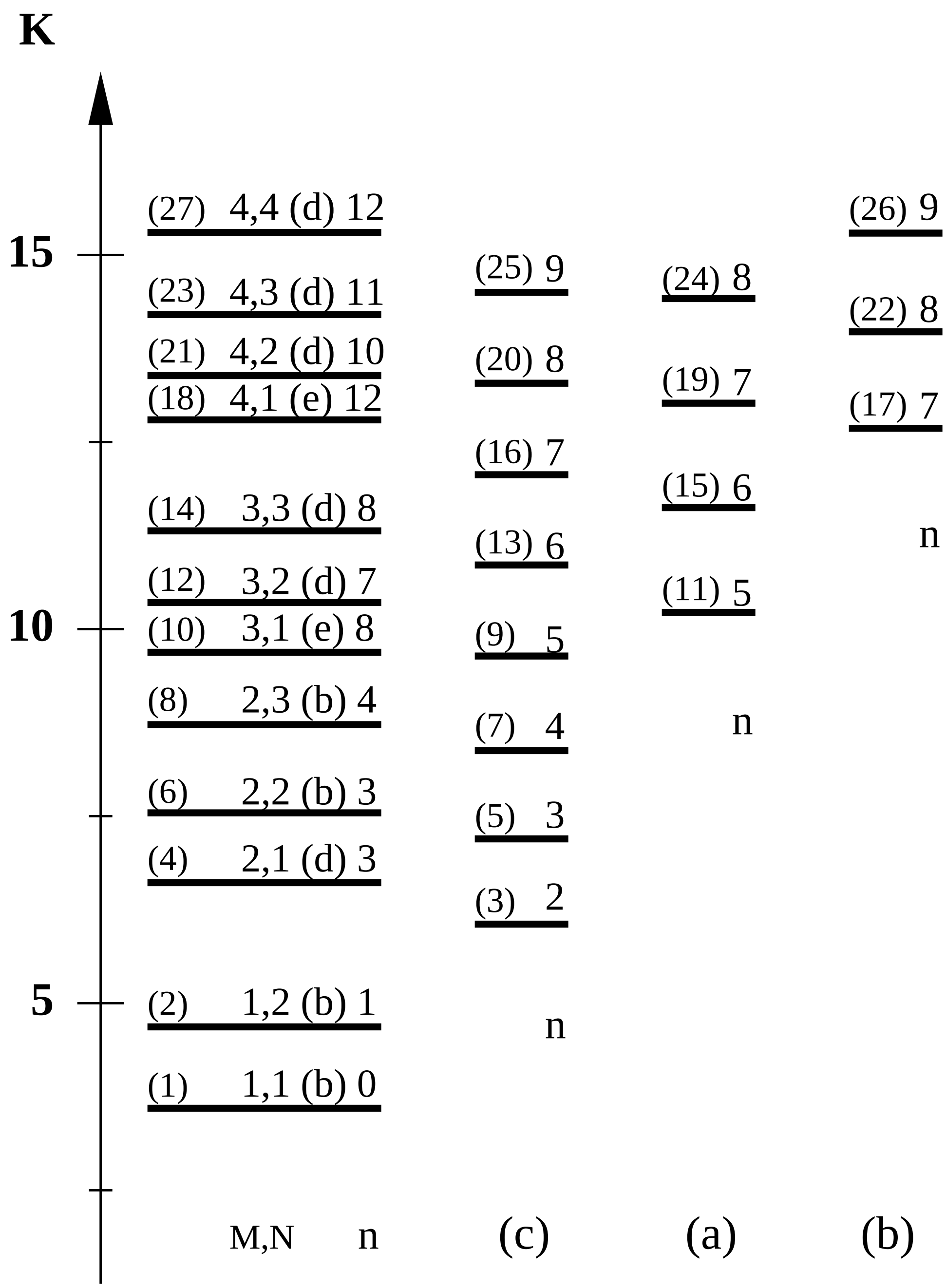


(a)
X=√5
θ
X1=0
(1)
(b)
X=1+√2
(1)
X1=0
(c)
X2=√3/2
π/6
(2)
X=3√3/2
X1=0
(1)
(d)
X2=√5/2
X=1+√5
X1=0
(1)
(2)
(e)
X2=2√10/3
X=1+√10
X1=0
(1)
(2)
(f)
(1)
X1=0
X2=1+1/√2
π/4
X=1+√2
(2)



Y
y
x
y
x
α_1
X
$y^{(2)}$
$x^{(2)}$
$y^{(1)}$
$x^{(1)}$



K
15
10
5
(27) 4,4 (d) 12
(23) 4,3 (d) 11
(21) 4,2 (d) 10
(18) 4,1 (e) 12
(14) 3,3 (d) 8
(12) 3,2 (d) 7
(10) 3,1 (e) 8
(8) 2,3 (b) 4
(6) 2,2 (b) 3
(4) 2,1 (d) 3
(2) 1,2 (b) 1
(1) 1,1 (b) 0
(25) 9
(20) 8
(16) 7
(13) 6
(9) 5
(7) 4
(5) 3
(3) 2
n
(24) 8
(19) 7
(15) 6
(11) 5
n
(26) 9
(22) 8
(17) 7
n
M,N n
(c)
(a)
(b)